Poisson structure of a modified Hunter–Saxton equation

2008 J. Phys. A: Math. Theor. 41 285207

(http://iopscience.iop.org/1751-8121/41/28/285207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 285207 (9pp) doi:10.1088/1751-8113/41/28/285207

Poisson structure of a modified Hunter–Saxton equation

Jonatan Lenells

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK

E-mail: J.Lenells@damtp.cam.ac.uk

Received 1 February 2008, in final form 1 May 2008
Published 24 June 2008
Online at stacks.iop.org/JPhysA/41/285207

Abstract

Both the well-known Korteweg–de Vries equation and the Hunter–Saxton equation with a linear u_x-term, $-u_{txx} = -2\omega u_x + 2u_x u_{xx} + u u_{xxx}$, $\omega > 0$, are bi-Hamiltonian in the sense that each equation is Hamiltonian with respect to two compatible, but distinct, Poisson brackets. We present a nonlinear change of variables which maps between the two bi-Poisson structures, giving a correspondence between the infinite hierarchies of conservation laws and commuting flows associated with the equations. In particular, under this change of variables this (modified) Hunter–Saxton equation can be viewed as a member of the KdV hierarchy.

PACS numbers: 02.30.Ik, 02.30.Jr
Mathematics Subject Classification: 37K10, 53D17

1. Introduction

Since the discovery that the KdV equation

$$Q_t - \tfrac{3}{2} Q Q_y + \tfrac{1}{4} Q_{yyy} = 0, \qquad y \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

can be viewed as an infinite-dimensional completely integrable Hamiltonian system [9, 19] (see also [7]), several other models for nonlinear wave propagation have also been found to admit a similar structure. Among the properties normally exhibited by these integrable PDEs are the existence of two compatible, but distinct, Hamiltonian structures, an infinite number of conserved quantities, a Lax pair formulation and soliton solutions. All of these characteristics are found in the equation

$$-u_{txx} = -2\omega u_x + 2u_x u_{xx} + u u_{xxx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

where $\omega \in \mathbb{R}$ is a constant parameter. For $\omega = 0$ equation (1.2) was derived by Hunter and Saxton as a model for the propagation of nonlinear waves in the director field for a nematic liquid crystal [11], whereas for $\omega > 0$ it is a special case of an equation for short capillary-gravity waves [17]. Some aspects of (1.2) resemble those of the Camassa–Holm equation [4]

$$u_t - u_{txx} + 3uu_x = -2\omega u_x + 2u_x u_{xx} + uu_{xxx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.3}$$

which also belongs to the class of integrable nonlinear wave equations with a bi-Hamiltonian formulation [8, 16]. In particular, in addition to being integrable for a large class of initial data, both (1.2) and (1.3) present smooth solutions that develop singularities in finite time, cf [3, 5, 10]. Just like (1.3) is studied both when $\omega = 0$ and when $\omega > 0$, and the two cases are qualitatively different in several respects (e.g. the peaked solitons exist only when $\omega = 0$, cf [14], and the location and the number of eigenvalues of the associated isospectral problems are very different [6, 13]), equation (1.2) with $\omega > 0$ has some interesting features not shared by the Hunter–Saxton equation. For example, whereas the Hunter–Saxton equation admits no travelling wave solutions, equation (1.2) exhibits both smooth as well as cusped travelling waves whenever $\omega > 0$. The cusped soliton solutions of (1.2) were studied using tools of algebraic and complex geometry in [1], while the hierarchy associated with equation (1.2) was considered in [2].

The bi-Hamiltonian character of (1.1) and (1.2) implies that each equation is Hamiltonian with respect to two compatible, but distinct, Poisson brackets. In this paper we point out that there is a nonlinear change of variables which transforms the two Poisson brackets associated with (1.2) into those of (1.1), giving a correspondence between the infinite hierarchies of conservation laws and commuting flows associated with the equations. In particular, under this change of variables equation (1.2) is mapped to the first member of the negative KdV hierarchy. The transformation is only well defined for solutions u of (1.2) satisfying $-u_{xx}+\omega > 0$, which means that it is not applicable to the case ($\omega = 0$) of the original Hunter–Saxton equation. However, for $\omega > 0$, this condition is fulfilled by all solutions with u_{xx} sufficiently small, which is the class of functions we will restrict ourselves to.

The bi-Hamiltonian formulations of (1.1) and (1.2) are presented in section 2. Section 3 contains the definition of the nonlinear transformation and describes how it relates the Poisson structures, while the final section deals with the correspondence of the two hierarchies.

2. bi-Hamiltonian formulations

Equation (1.2) has the bi-Hamiltonian formulation

$$m_t = X_{H_1}^{\mathcal{B}_1}[m] = X_{H_2}^{\mathcal{B}_2}[m],$$

where $m = -u_{xx}$, and

$$X_{H_1}^{\mathcal{B}_1} = \mathcal{B}_1 \frac{\delta H_1}{\delta m}, \qquad X_{H_2}^{\mathcal{B}_2} = \mathcal{B}_2 \frac{\delta H_2}{\delta m}$$

are the Hamiltonian vector fields corresponding to the functionals

$$H_1 = \frac{1}{2}\int um \, \mathrm{d}x, \qquad H_2 = \frac{1}{2}\int \left(uu_x^2 + 2\omega u^2\right) \mathrm{d}x,$$

with respect to the Poisson brackets

$$\{h_1, h_2\}_{\mathcal{B}_1} = \int \frac{\delta h_1}{\delta m}\mathcal{B}_1 \frac{\delta h_2}{\delta m}\, \mathrm{d}x, \qquad \{h_1, h_2\}_{\mathcal{B}_2} = \int \frac{\delta h_1}{\delta m}\mathcal{B}_2 \frac{\delta h_2}{\delta m}\, \mathrm{d}x,$$
$$\mathcal{B}_1 = -((m+\omega)D_x + D_x(m+\omega)), \qquad \mathcal{B}_2 = D_x^3.$$

Similarly, the KdV equation (1.1) can be written as

$$Q_t = X_{G_1}^{\mathcal{C}_1}[Q] = X_{G_2}^{\mathcal{C}_2}[Q],$$

where

$$X_{G_1}^{\mathcal{C}_1} = \mathcal{C}_1 \frac{\delta G_1}{\delta Q}, \qquad X_{G_2}^{\mathcal{C}_2} = \mathcal{C}_2 \frac{\delta G_2}{\delta Q}$$

are the Hamiltonian vector fields corresponding to the functionals

$$G_1 = \frac{1}{4}\int Q^2\,\mathrm{d}y, \qquad G_2 = \frac{1}{4}\int \left(Q^3 + \frac{1}{2}Q_y^2\right)\mathrm{d}y,$$

with respect to the Poisson brackets

$$\{g_1, g_2\}_{\mathcal{C}_1} = \int \frac{\delta g_1}{\delta Q}\mathcal{C}_1 \frac{\delta g_2}{\delta Q}\,\mathrm{d}y, \qquad \{g_1, g_2\}_{\mathcal{C}_2} = \int \frac{\delta g_1}{\delta Q}\mathcal{C}_2 \frac{\delta g_2}{\delta Q}\,\mathrm{d}y,$$

$$\mathcal{C}_1 = -\frac{1}{2}D_y^3 + QD_y + D_y Q, \qquad \mathcal{C}_2 = D_y.$$

The recursive definitions (see [20] for more background),

$$\mathcal{B}_1 \frac{\delta H_n}{\delta m} = \mathcal{B}_2 \frac{\delta H_{n+1}}{\delta m}, \qquad \mathcal{C}_1 \frac{\delta G_n}{\delta Q} = \mathcal{C}_2 \frac{\delta G_{n+1}}{\delta Q}, \qquad n \in \mathbb{Z}, \tag{2.1}$$

yield infinite sequences of functionals

$$\ldots, H_{-1}, H_0, H_1, H_2, \ldots, \qquad \ldots, G_{-1}, G_0, G_1, G_2, \ldots,$$

conserved under the flows of (1.2) respectively (1.1). The corresponding hierarchies consist of the flows,

$$m_t = X_n^{\mathcal{B}_2}[m], \qquad Q_t = X_n^{\mathcal{C}_2}[Q], \qquad n \in \mathbb{Z},$$

generated by these Hamiltonians, where

$$X_n^{\mathcal{B}_2}[m] = \mathcal{B}_2 \frac{\delta H_n}{\delta m}, \qquad X_n^{\mathcal{C}_2}[Q] = \mathcal{C}_2 \frac{\delta G_n}{\delta m}.$$

3. Correspondence of Poisson structures

The equations in the KdV hierarchy describe commuting flows on the infinite-dimensional level surfaces in the space of functions $Q(y)$ obtained by fixing the values of the G_n's, while the hierarchy associated with (1.2) consists of commuting flows on surfaces in the space of functions $m(x)$. In the following we will construct a nonlinear map $\Phi : m(x) \mapsto Q(y)$ which preserves the bi-Poisson structures and maps the commuting flows of equation (1.2) into those of the KdV equation, that is, under this transformation the two hierarchies are equivalent.

3.1. Liouville transformation

The motivation for the change of variables derives from the Lax pair formulations of the two equations. Assuming $m + \omega > 0$, the Liouville transformation,

$$y = D_x^{-1}\sqrt{m+\omega}, \qquad \phi(y) = (m(x)+\omega)^{1/4}\psi(x), \tag{3.1}$$

converts the isospectral problem for (1.2),

$$\psi_{xx} + \lambda(m+\omega)\psi = 0,$$

into the isospectral problem for the KdV equation,

$$-\phi_{yy} + Q(y)\phi = \mu\phi,$$

where $\mu = -\lambda$ and

$$Q(y) = -\frac{((m(x)+\omega)^{-1/4})_{xx}}{(m(x)+\omega)^{3/4}}. \tag{3.2}$$

Here

$$\left(D_x^{-1}\sqrt{m+\omega}\right)(x) = x\sqrt{\omega} + \int_{-\infty}^{x} (\sqrt{m(\xi)+\omega} - \sqrt{\omega})\,\mathrm{d}\xi$$

in the case on the line, and

$$\left(D_x^{-1}\sqrt{m+\omega}\right)(x) = \int_0^x \sqrt{m(\xi)+\omega}\,\mathrm{d}\xi$$

in the periodic case. Note that $H_{-1} = \int(\sqrt{m+\omega} - \sqrt{\omega})\,\mathrm{d}x$ is the first conservation law in the negative hierarchy for (1.2) (see table 1), so that if $m(x)$ is periodic, then the period $\int\sqrt{m+\omega}\,\mathrm{d}x$ of $Q(y)$ is preserved under all flows in the hierarchy.

Let us comment on the requirement

$$m + \omega = -u_{xx} + \omega > 0. \tag{3.3}$$

The constructions presented in the following sections formally also apply to the case $\omega = 0$ corresponding to the Hunter–Saxton equation. If $\omega = 0$, however, condition (3.3) can never be fulfilled in the periodic setting, and, under realistic boundary conditions, it is also violated in the case on the line. On the other hand, for $\omega > 0$ and solutions u with u_{xx} small, (3.3) is seen to hold. In particular, all periodic smooth solitons of (1.2) satisfy (3.3) and are thus covered by our approach c.f. [15].

3.2. Poisson correspondence

Define Φ as the map taking $m(x)$ to the function $Q(y)$ defined by (3.1) and (3.2). This transformation will provide the correspondence between the two hierarchies. Before stating the main result, we only need to make one more observation. Although the simplicity of the Hamiltonian operators $\mathcal{C}_1$ and $\mathcal{C}_2$ make the Poisson brackets $\{\cdot,\cdot\}_{\mathcal{C}_1}$ and $\{\cdot,\cdot\}_{\mathcal{C}_2}$ introduced in section 2 natural choices, the KdV equation can just as well be written as a bi-Hamiltonian system in terms of the Poisson structures

$$\{g_1, g_2\}_1 = -\frac{1}{2}\int \frac{\delta g_1}{\delta Q}\mathcal{C}_1\mathcal{C}_2^{-1}\mathcal{C}_1\frac{\delta g_2}{\delta Q}$$

and

$$\{g_1, g_2\}_2 = -\frac{1}{2}\int \frac{\delta g_1}{\delta Q}\mathcal{C}_1\mathcal{C}_2^{-1}\mathcal{C}_1\mathcal{C}_2^{-1}\mathcal{C}_1\frac{\delta g_2}{\delta Q}$$

Since

$$\mathcal{C}_2^{-1}\mathcal{C}_1\frac{\delta G_n}{\delta Q} = \frac{\delta G_{n+1}}{\delta Q},$$

the KdV equation is bi-Hamiltonian with respect to $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ with Hamiltonians $-2G_0$ respectively $-2G_{-1}$. This change of Poisson brackets only amounts to a renumbering of the commuting flows in the hierarchy.

Theorem 1. *The transformation Φ maps the two Poisson structures for equation (1.2) into those of the KdV equation. More precisely, for any two functionals $g_1[Q]$ and $g_2[Q]$ it holds that*

$$\{g_1\circ\Phi, g_2\circ\Phi\}_{\mathcal{B}_1} = \{g_1, g_2\}_1\circ\Phi \tag{3.4}$$

and

$$\{g_1 \circ \Phi, g_2 \circ \Phi\}_{\mathcal{B}_2} = \{g_1, g_2\}_2 \circ \Phi. \tag{3.5}$$

To prove theorem 1 we first note that use of the identities

$$D_x = \sqrt{m+\omega} D_y, \qquad D_x \frac{1}{\sqrt{m+\omega}} = \frac{1}{\sqrt{m+\omega}} D_x - \frac{m_x}{2(m+\omega)^{3/2}}$$

shows that the Hamiltonian operators for KdV and HS are related by

$$\mathcal{B}_1 = -2(m+\omega)\mathcal{C}_2\sqrt{m+\omega}, \qquad \mathcal{B}_2 = -2(m+\omega)\mathcal{C}_1\sqrt{m+\omega}. \tag{3.6}$$

Moreover, we claim that for any functional $g[Q]$ it holds that

$$\frac{\delta(g\circ\Phi)}{\delta m} = -\frac{1}{2\sqrt{m+\omega}}\mathcal{C}_2^{-1}\mathcal{C}_1 \frac{\delta g}{\delta Q}\bigg|_{Q=\Phi[m]}. \tag{3.7}$$

If we temporarily assume this to be true, it follows from (3.6) that

$$\begin{aligned}\{g_1 \circ \Phi, g_2 \circ \Phi\}_{\mathcal{B}_1} &= \int \frac{\delta(g_1\circ\Phi)}{\delta m}\mathcal{B}_1\frac{\delta(g_2\circ\Phi)}{\delta m}\,\mathrm{d}x \\ &= -\frac{1}{2}\int \frac{\delta g_1}{\delta Q}\mathcal{C}_1\mathcal{C}_2^{-1}\mathcal{C}_1\frac{\delta g_2}{\delta Q}\mathrm{d}y \\ &= \{g_1, g_2\}_1 \circ \Phi.\end{aligned}$$

This proves (3.4) and a similar computation yields (3.5). Here and in the following we assume that the function spaces have been chosen so that $\mathcal{C}_2^{-1} = D_y^{-1}$ is skew-adjoint (in a situation where this is not true the terms produced by constants of integration have to be incorporated by hand).

To establish the claim (3.7) we need to compute the tangent map of the transformation Φ. Let

$$F[m] = Q = -\frac{((m(x)+\omega)^{-1/4})_{xx}}{(m(x)+\omega)^{3/4}}. \tag{3.8}$$

The derivative of $Q(t, y)$ with respect to t holding x fixed is

$$Q_y(t,y)\frac{\partial y}{\partial t} + Q_t(t,y) = Q_y D_x^{-1}\frac{m_t}{2\sqrt{m+\omega}} + Q_t.$$

In view of (3.8) this equals $F'[m]\cdot m_t$, where

$$F'[m] = -\frac{Q}{m+\omega} - \frac{((m+\omega)^{-1/4})_{xx}}{4(m+\omega)^{7/4}} + \frac{1}{4(m+\omega)^{3/4}}D_x^2\frac{1}{(m+\omega)^{5/4}}.$$

Hence, if $m_t = X[m]$ and $Q_t = Y[Q]$, then

$$Y[Q] = \left(F'[m] - Q_y D_x^{-1}\frac{1}{2\sqrt{m+\omega}}\right)X[m],$$

that is, the tangent map of Φ at m is

$$T_m\Phi = F'[m] - Q_y D_x^{-1}\frac{1}{2\sqrt{m+\omega}}.$$

We can rewrite this by noticing that

$$F'[m] = \left(\frac{1}{4}D_y^2 - Q\right)\frac{1}{m+\omega} = -\frac{1}{2}\mathcal{C}_1\mathcal{C}_2^{-1}\frac{1}{m+\omega} + Q_y D_x^{-1}\frac{1}{2\sqrt{m+\omega}},$$

Table 1. Conservation laws of mHS.

n	H_n	$\frac{\delta H_n}{\delta m}$
2	$\frac{1}{2}\int\left(uu_x^2+2\omega u^2\right)\,\mathrm{d}x$	$D_x^{-2}\left(\frac{1}{2}u_x^2+uu_{xx}-2\omega u\right)$
1	$\frac{1}{2}\int um\,\mathrm{d}x$	u
0	$\int m\,\mathrm{d}x$	1
-1	$\int(\sqrt{m+\omega}-\sqrt{\omega})\,\mathrm{d}x$	$\frac{1}{2\sqrt{m+\omega}}$
-2	$-\frac{1}{16}\int\frac{m_x^2}{(m+\omega)^{5/2}}\,\mathrm{d}x$	$\frac{m_{xx}}{8(m+\omega)^{5/2}}-\frac{5m_x^2}{32(m+\omega)^{7/2}}$

Table 2. Conservation laws of KdV.

n	G_n	$\frac{\delta G_n}{\delta Q}$
2	$\frac{1}{4}\int\left(Q^3+\frac{1}{2}Q_y^2\right)\mathrm{d}y$	$\frac{3}{4}Q^2-\frac{1}{4}Q_{yy}$
1	$\frac{1}{4}\int Q^2\,\mathrm{d}y$	$\frac{1}{2}Q$
0	$\frac{1}{2}\int Q\,\mathrm{d}y$	$\frac{1}{2}$
-1	$-\frac{1}{2}\int\left(1-\frac{\sqrt{\omega}}{\sqrt{m+\omega}}\right)\,\mathrm{d}y$	$\sqrt{m+\omega}$
-2	$-\frac{1}{2}\int\frac{m}{\sqrt{m+\omega}}\,\mathrm{d}y$	$u\sqrt{m+\omega}$

and so

$$T_m\Phi=-\frac{1}{2}\mathcal{C}_1\mathcal{C}_2^{-1}\frac{1}{m+\omega}. \tag{3.9}$$

Now, for a functional $g[Q]$, we have by definition of the variational derivative

$$\int\frac{\delta(g\circ\Phi)}{\delta m}\varphi\,\mathrm{d}x=\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}g[\Phi[m+\epsilon\varphi]]=\int\frac{\delta g}{\delta Q}T\Phi\cdot\varphi\,\mathrm{d}y.$$

It follows that

$$\frac{\delta(g\circ\Phi)}{\delta m}=\sqrt{m+\omega}T^*\Phi\cdot\frac{\delta g}{\delta Q}, \tag{3.10}$$

where $T^*\Phi$ is the adjoint of $T\Phi$ with respect to $\int\mathrm{d}y$, and the factor $\sqrt{m+\omega}$ arises because the two dual spaces are identified using the inner products $\int\mathrm{d}x$ respectively $\int\mathrm{d}y$ related by $\mathrm{d}y=\sqrt{m+\omega}\,\mathrm{d}x$. We know from (3.9) and the skew-adjointness of the Hamiltonian operators that $T^*\Phi=-\frac{1}{2(m+\omega)}\mathcal{C}_2^{-1}\mathcal{C}_1$. Hence the claim (3.7) follows from (3.10).

4. Correspondence of hierarchies

The correspondence of Poisson structures described in the previous section implies that the hierarchies of equations associated with KdV and the modified Hunter–Saxton (1.2) (abbreviated mHS in this section) are also related. The first few conservation laws in the two hierarchies are presented in tables 1 and 2. Observe, however, that since the Hamiltonian operators contain derivative operators, there is an integration constant ambiguity at each step of the construction of the ladder of conservation laws according to (2.1). In tables 1 and 2 these constants have been chosen so as to give the simplest possible expressions. Furthermore, note that when descending the negative KdV hierarchy, or, equivalently, when climbing the positive hierarchy for (1.2), increasingly nonlocal conservation laws appear, e.g. $H_0=\int m\,\mathrm{d}x$ is local in m; $H_1=\frac{1}{2}\int mu\,\mathrm{d}x$ and $H_2=\frac{1}{2}\int\left(uu_x^2+2\omega u^2\right)\mathrm{d}x$ are nonlocal in m, but become local when written in terms of $u = -D_x^{-2}m$; H_3 is nonlocal also in u and requires another inversion of D_x^2, etc. We do not consider in this paper the extent to which our computations can be justified in a rigorous analytical framework, but are content with pointing out the general structure of the correspondence and relating some of the simplest flows in the hierarchies (see [12, 18] for some further remarks about analytic details in similar situations).

4.1. Flows

The flow of equation (1.2),

$$m_t = \mathcal{B}_1 \frac{\delta H_1}{\delta m} = -2\omega u_x + 2u_x u_{xx} + u u_{xxx},$$

is mapped under the transformation Φ to the flow

$$Q_t = T\Phi \cdot \mathcal{B}_1 \frac{\delta H_1}{\delta m}.$$

Using expression (3.9) for $T\Phi$ and the first relation in (3.6) to replace $\mathcal{B}_1$ by $-2(m+\omega)\mathcal{C}_2\sqrt{m+\omega}$, this becomes

$$Q_t = \mathcal{C}_1 \mathcal{C}_2^{-1} \mathcal{C}_2 \sqrt{m+\omega} \frac{\delta H_1}{\delta m}.$$

Furthermore, from tables 1 and 2 we see that $\sqrt{m+\omega}\frac{\delta H_1}{\delta m} = u\sqrt{m+\omega} = \frac{\delta G_{-2}}{\delta Q}$, and so the evolution equation for Q is

$$Q_t = \mathcal{C}_1 \frac{\delta G_{-2}}{\delta Q},$$

which is the first member of the negative KdV hierarchy[1]. This shows that equation (1.2), which is the second flow in the positive mHS hierarchy, corresponds to the first flow in the negative KdV hierarchy.

Similarly, using the relations

$$\begin{aligned}
\frac{\delta H_{-2}}{\delta m} &= \frac{1}{2\sqrt{m+\omega}} Q = \frac{1}{\sqrt{m+\omega}} \frac{\delta G_1}{\delta Q}, \\
\frac{\delta H_{-1}}{\delta m} &= \frac{1}{2\sqrt{m+\omega}} = \frac{1}{\sqrt{m+\omega}} \frac{\delta G_0}{\delta Q}, \\
\frac{\delta H_0}{\delta m} &= 1 = \frac{1}{\sqrt{m+\omega}} \frac{\delta G_{-1}}{\delta Q},
\end{aligned}$$

we may deduce further correspondences between the flows in the two hierarchies. If we denote flow number n in the hierarchy associated with the modified Hunter–Saxton equation (1.2) by $(\mathrm{mHS})_n$ and the nth flow for the KdV hierarchy by $(\mathrm{KdV})_n$, the result can be presented as in table 3.

More generally, it follows that (at least modulo the problem of nonlocality and the choice of constants of integration mentioned above) m evolves according to $(\mathrm{mHS})_n$,

$$m_t = \mathcal{B}_2 \frac{\delta H_n}{\delta m} = \mathcal{B}_1 \frac{\delta H_{n-1}}{\delta m},$$

if and only if Q evolves according to $(\mathrm{KdV})_{-n+1}$,

$$Q_t = \mathcal{C}_2 \frac{\delta G_{-n+1}}{\delta m} = \mathcal{C}_1 \frac{\delta G_{-n}}{\delta m}.$$

This establishes the correspondence of the flows of the two hierarchies.

[1] Here we have assumed that $\mathcal{C}_2^{-1}\mathcal{C}_2 \frac{\delta G_{-2}}{\delta Q} = \frac{\delta G_{-2}}{\delta Q}$; otherwise a term arising from the integration constant has to be included.

Table 3. Correspondence of hierarchies.

mHS flow	Equation for m	Equation for Q	KdV flow
$(\mathrm{mHS})_2$	$m_t = -2\omega u_x + 2u_x u_{xx} + u u_{xxx}$	$Q_t = \frac{m_x}{2(m+\omega)}$	$(\mathrm{KdV})_{-1}$
$(\mathrm{mHS})_1$	$m_t = m_x$	$Q_t = 0$	$(\mathrm{KdV})_0$
$(\mathrm{mHS})_0$	$m_t = 0$	$Q_t = \frac{1}{2}Q_y$	$(\mathrm{KdV})_1$
$(\mathrm{mHS})_{-1}$	$m_t = -mQ_y$	$Q_t = \frac{3}{2}QQ_y - \frac{1}{4}Q_{yyy}$	$(\mathrm{KdV})_2$

4.2. *Conservation laws*

What is the correspondence between the conservation laws themselves? Let $g[Q]$ be a functional of Q. From theorem 1 we infer that $T\Phi$ maps the Hamiltonian flow induced by $(g\circ\Phi)[m]$ with respect to $\{\cdot,\cdot\}_{\mathcal{B}_1}$ into the Hamiltonian flow of $g[Q]$ with respect to $\{\cdot,\cdot\}_1$. In other words,

$$m_t = \mathcal{B}_1\frac{\delta(g\circ\Phi)}{\delta m} \tag{4.1}$$

maps to

$$Q_t = -\frac{1}{2}\mathcal{C}_1\mathcal{C}_2^{-1}\mathcal{C}_1\frac{\delta g}{\delta Q}.$$

Choosing g to be the conservation law G_{-n-1} for KdV, we can write the equation for Q as

$$Q_t = -\frac{1}{2}\mathcal{C}_1\mathcal{C}_2^{-1}\mathcal{C}_1\frac{\delta G_{-n-1}}{\delta Q} = -\frac{1}{2}\mathcal{C}_1\frac{\delta G_{-n}}{\delta Q}.$$

But we know from above that the corresponding flow for m is

$$m_t = -\frac{1}{2}\mathcal{B}_1\frac{\delta H_{n-1}}{\delta m}.$$

Comparing this with (4.1), we deduce that

$$\frac{\delta(G_{-n-1}\circ\Phi)}{\delta m} = -\frac{1}{2}\frac{\delta H_{n-1}}{\delta m},$$

i.e., the conservation laws are related by

$$H_n[m] = -2G_{-n-2}[Q], \qquad n\in\mathbb{Z}.$$

5. Concluding remarks

In this paper we have presented with the help of a Liouville transformation a nonlinear correspondence between two soliton hierarchies: the hierarchy of the modified Hunter–Saxton equation (1.2) and the KdV hierarchy. Under this transformation, the positive part of the KdV hierarchy is identified with the negative part of the Hunter–Saxton hierarchy and vice versa. The modified Hunter–Saxton equation corresponds to the first negative member of the KdV hierarchy. Let us point out that there exists a similar transformation relating the hierarchies of the Camassa–Holm and KdV equations, so that equation (1.3) can also be regarded as the first negative member of the KdV hierarchy [18]. In particular, applying these two transformations in sequence provides a map taking solutions of (1.2) to solutions of (1.3).

The approach is not applicable to the original Hunter–Saxton equation since the transformation is well defined only for solutions u of (1.2) satisfying $-u_{xx}+\omega > 0$, a condition that cannot be fulfilled under reasonable boundary conditions when $\omega = 0$. On the other hand, for $\omega > 0$, this condition is fulfilled by a large class of solutions for which u_{xx} is small. In particular, it can be shown that all periodic smooth solitons of (1.2) belong to this class, cf [15].

Acknowledgments

The author thanks Adrian Constantin for helpful remarks on a first version of the manuscript and acknowledges support from a Marie Curie Intra-European Fellowship. He is also grateful to the referees for valuable suggestions.

References

[1] Alber M S, Camassa R, Holm D D and Marsden J E 1994 The geometry of peaked solitons and billiard solutions of a class of integrable PDE's *Lett. Math. Phys.* **32** 137–51
[2] Baran H 2005 Can we always distinguish between positive and negative hierarchies? *J. Phys. A: Math. Gen.* **38** 301–6
[3] Bressan A and Constantin A 2005 Global solutions of the Hunter–Saxton equation *SIAM J. Math. Anal.* **37** 996–1026
[4] Camassa R and Holm D D 1993 An integrable shallow water equation with peaked solitons *Phys. Rev. Lett.* **71** 1661–4
[5] Constantin A and Escher J 1998 Wave breaking for nonlinear nonlocal shallow water equations *Acta Math.* **181** 229–43
[6] Constantin A and McKean H P 1999 A shallow water equation on the circle *Commun. Pure Appl. Math.* **52** 949–82
[7] Drazin P G and Johnson R S 1989 *Solitons: An introduction* (Cambridge: Cambridge University Press)
[8] Fuchssteiner B and Fokas A S 1981 Symplectic structures, their Bäcklund transformation and hereditary symmetries *Physica* D **4** 47–66
[9] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg–de Vries equation *Phys. Rev. Lett.* **19** 1095–7
[10] Holden H, Karlsen K H and Risebro N H 2007 Convergent difference schemes for the Hunter–Saxton equation *Math. Comput.* **76** 699–744
[11] Hunter J K and Saxton R 1991 Dynamics of director fields *SIAM J. Appl. Math.* **51** 1498–521
[12] Hunter J K and Zheng Y 1994 On a completely integrable nonlinear hyperbolic variational equation *Physica* D **79** 361–86
[13] Lenells J 2002 The scattering approach for the Camassa–Holm equation *J. Nonlinear Math. Phys.* **9** 389–93
[14] Lenells J 2005 Travelling wave solutions of the Camassa–Holm equation *J. Diff. Eq.* **217** 393–430
[15] Lenells J Periodic solitons of an equation for short capillary-gravity waves unpublished
[16] Magri F 1978 A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62
[17] Manna M A and Neveu A 2003 A singular integrable equation from short capillary-gravity waves (*Preprint* physics/0303085)
[18] McKean H P 2003 The Liouville correspondence between the Korteweg–de Vries and the Camassa–Holm hierarchies *Commun. Pure Appl. Math.* **56** 998–1015
[19] Miura R M, Gardner C S and Kruskal M D 1968 Korteweg–de Vries equation and generalizations: II. Existence of conservation laws and constants of motion *J. Math. Phys.* **9** 1204–9
[20] Olver P J 1993 *Application of Lie Groups to Differential Equations* 2nd edn (New York: Springer)